

LIFTING REPRESENTATIONS OF FINITE REDUCTIVE GROUPS II: EXPLICIT CONORMS

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ABSTRACT. Let k be a field, \tilde{G} a connected reductive k -quasisplit group, Γ a finite group that acts on \tilde{G} via k -automorphisms satisfying a quasi-semisimplicity condition, and G the connected part of the group of Γ -fixed points of \tilde{G} , also assumed k -quasisplit. In an earlier work, the authors constructed a canonical map $\hat{\mathcal{N}}$ from the set of stable semisimple conjugacy classes in the dual $G^*(k)$ to the set of such classes in $\tilde{G}^*(k)$. We describe several situations where $\hat{\mathcal{N}}$ can be refined to an explicit function on points, or where it factors through such a function.

0. INTRODUCTION

Suppose that k is a field, \tilde{G} is a connected reductive k -group, Γ is a finite group that acts on \tilde{G} via k -automorphisms, and G is the connected part of the group \tilde{G}^Γ of fixed points. Assume that Γ fixes some pair (\tilde{B}, \tilde{T}) consisting of a Borel subgroup $\tilde{B} \subseteq \tilde{G}$ and a maximal torus $\tilde{T} \subseteq \tilde{B}$. We need not assume that \tilde{B} or \tilde{T} is defined over k . In an earlier work [1], we showed that G is a reductive k -group, and that if G is k -quasisplit then we can indeed choose \tilde{T} to be defined over k . If \tilde{G} is also k -quasi-split, then one can form the duals \tilde{G}^* and G^* , and we showed that the action of Γ induces a canonical k -morphism $\hat{\mathcal{N}}_{\tilde{G}, \Gamma}$ from the variety of semisimple geometric conjugacy classes in G^* to the analogous variety for \tilde{G}^* . (We omit one or both of the subscripts when they are clear from context.) Moreover, this map specializes to give a map, here denoted $\hat{\mathcal{N}}_{\tilde{G}, \Gamma}^{\text{st}}$, from the set of stable (in the sense of Kottwitz [3]) semisimple conjugacy classes in $G^*(k)$ to the analogous set for $\tilde{G}^*(k)$.

In the special case where k is finite, we have that G and \tilde{G} are automatically k -quasisplit, and stable and rational conjugacy coincide. Since $G^*(k)$ -conjugacy classes in $G^*(k)$ parametrize collections of irreducible representations of $G(k)$, and similarly for $\tilde{G}(k)$, one obtains a lifting of such collections from $G(k)$ to $\tilde{G}(k)$.

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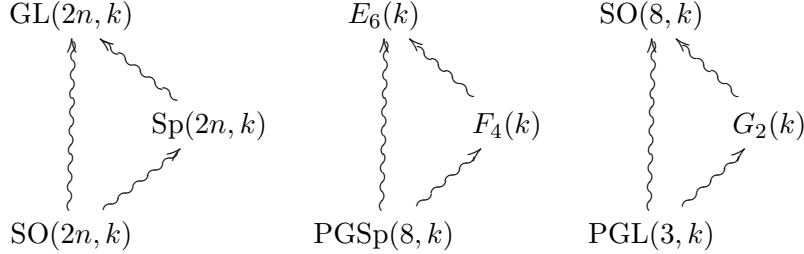
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The description of the map $\hat{\mathcal{N}}$ in [1] is explicit in some sense. Given semisimple $s \in G^*(k)$, choose a maximal k -torus $T^* \subseteq G^*$ such that $s \in T^*(k)$. It is a simple matter to construct a corresponding k -torus $\tilde{T}^* \subseteq \tilde{G}^*$, and a k -homomorphism $\hat{\mathcal{N}}_{T^*}: T^* \rightarrow \tilde{T}^*$. Then $\hat{\mathcal{N}}^{\text{st}}(s)$ is the stable class containing $\hat{\mathcal{N}}_{T^*}(s)$. In particular, the choices of T^* and \tilde{T}^* don't matter.

However, there are several situations where one can make $\hat{\mathcal{N}}$ even more explicit. For example, sometimes $\hat{\mathcal{N}}$ can be expressed as a composition of potentially simpler functions, and sometimes either $\hat{\mathcal{N}}$ or one of these factors can be refined to an actual morphism of groups.

A particularly interesting case is the following. Suppose (to simplify the present discussion) that \tilde{G} is almost simple, Γ is cyclic, and no nontrivial element of Γ acts via inner automorphisms. Then we can construct another action of Γ that belongs to the same inner class as the original, but that also fixes a pinning for $(\tilde{G}, \tilde{B}, \tilde{T})$. Let \underline{G} denote the connected part of the fixed point group of this latter action. Then there is a natural embedding $G^* \hookrightarrow \underline{G}^*$, and $\hat{\mathcal{N}}$ factors through it (Proposition 7.7). For example, suppose Γ acts on $\text{GL}(2n)$ via an involution, and $G = \text{SO}(2n)$. Then $\underline{G} = \text{Sp}(2n)$, and our result says that our lifting of (families of) representations from $\text{SO}(2n, k)$ to $\text{GL}(2n, k)$ (k finite) must factor through a lifting from $\text{SO}(2n, k)$ to $\text{Sp}(2n, k)$. Other liftings have analogous factorizations:



Another case occurs when G is contained in a proper Levi subgroup \tilde{L} of \tilde{G} . Then one can choose \tilde{L} to be defined over k and Γ -invariant, and $\hat{\mathcal{N}}_{\tilde{G}}$ is the composition of $\hat{\mathcal{N}}_{\tilde{L}}$ with the map on conjugacy classes induced by an embedding $\tilde{L}^* \subseteq \tilde{G}^*$ (Proposition 2.3). Even if G is not contained in a proper Levi subgroup of \tilde{G} , a similar conclusion will often follow (Remark 2.4). Moreover, we can often still describe $\hat{\mathcal{N}}(s)$ for a particular semisimple $s \in G^*(k)$ by replacing \tilde{G} by a smaller group \tilde{L} as above, but \tilde{L} could depend on s (Proposition 2.7).

Given a subnormal series for Γ , one can express the map $\hat{\mathcal{N}}$ as a composition of maps associated to the subquotients of Γ (Proposition 3.1).

From the point of view of base change, an important case is where \tilde{G} is a direct product of copies of G , and Γ acts by transitive permutation of the coordinates. In this case, $\hat{\mathcal{N}}$ arises from the diagonal embedding $G^* \hookrightarrow \tilde{G}^*$,

composed with a power map (Proposition 4.2), and the power map doesn't appear when Γ acts simply on the factors (Corollary 4.3).

Returning to the general situation, suppose we have a Γ -invariant k -isogeny $\tilde{G} \rightarrow \tilde{G}'$. Defining G' , G'^* , and \tilde{G}'^* as above, we have a conorm function \hat{N}' that maps semisimple classes in G'^* to those in \tilde{G}'^* , and one can express \hat{N} in terms of \hat{N}' (Corollary 5.2). This allows us to replace \tilde{G} by a direct product of a torus and a collection of almost k -simple groups. The action of Γ on such a product could be very complicated. But in many situations, including the ones easiest to describe, we can use the product decomposition and Proposition 3.1 to reduce the problem of understanding \hat{N} to the case where \tilde{G} is absolutely almost-simple. This is carried out in §6.

If Γ acts via inner automorphisms, then G will usually be contained in a proper Levi subgroup of \tilde{G} , and Proposition 2.3 applies. If there is no such containment, then \hat{N} generally cannot be refined to a function of points.

We assume that \tilde{G} and G are k -quasisplit only in order to apply the main theorems of [1, §6]. However, as remarked there, weaker hypotheses suffice, a matter that we will consider elsewhere.

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1. BASIC PROPERTIES OF THE CONORM FUNCTION

Let k be a field. For any reductive k -group H , denote by H° the connected component of the identity in H . If Γ is any finite group that acts on H via k -automorphisms, let H^Γ denote the group of Γ fixed points in H . If T is a maximal k -torus of H , denote by $\Phi(H, T)$ the system of roots of T in H , and by $W(H, T)$ the Weyl group of T in H . Let $\mathbf{X}^*(T)$ and $\mathbf{X}_*(T)$ respectively denote the character and cocharacter modules of T , and let $V^*(T) = \mathbf{X}^*(T) \otimes \mathbb{Q}$ and $V_*(T) = \mathbf{X}_*(T) \otimes \mathbb{Q}$.

For a fixed $W(H, T)$ -invariant inner product on $V^*(T)$, we will say that a root $\alpha \in \Phi(H, T)$ is *short* (resp. *long*) if its length with respect to this inner product is minimal (maximal) among all roots in the irreducible subsystem of $\Phi(H, T)$ containing α .

Any homomorphism $f: T \rightarrow T'$ of tori determines maps $f^*: \mathbf{X}^*(T') \rightarrow \mathbf{X}^*(T)$ and $f_*: \mathbf{X}_*(T) \rightarrow \mathbf{X}_*(T')$, and hence maps $V^*(T') \rightarrow V^*(T)$ and $V_*(T) \rightarrow V_*(T')$ that we will also denote by f^* and f_* , respectively.

If H is k -quasisplit, then there is reductive k -group H^* (unique up to k -isomorphism) in k -duality with H . That is, there are maximal k -tori $T \subseteq H$ and $T^* \subseteq H^*$ and a $\text{Gal}(k)$ -equivariant isomorphism $X^*(T) \rightarrow X_*(T^*)$ that respects the root and coroot systems of T (resp. T^*) in H (resp. H^*), as in [1, §6]. In this case, we say that T and T^* are in k -duality (a notion that depends on the ambient groups H and H^*). We will refer to δ as a *duality map*. There is a natural correspondence between the stable conjugacy classes

of maximal k -tori in H and those in H^* . Moreover, if $S \subseteq H$ and $S^* \subseteq H^*$ are corresponding tori, then S and S^* are in k -duality (see [1, Prop. 6.4]).

From now on, suppose that \tilde{G} is a connected reductive k -group, and Γ is a finite group that acts on \tilde{G} via k -automorphisms that preserve a Borel subgroup of \tilde{G} and a maximal torus in that Borel subgroup. Let $G = (\tilde{G}^\Gamma)^\circ$.

- Proposition 1.1.** (a) G is a reductive k -group.
 (b) For every Borel-torus pair (\tilde{B}, \tilde{T}) in \tilde{G} preserved by Γ , we have that $(\tilde{B}^\Gamma, (\tilde{T}^\Gamma)^\circ)$ is a Borel-torus pair for G .
 (c) Let T be a maximal torus in G , and let $\tilde{T} = C_{\tilde{G}}(T)$. Then \tilde{T} is a maximal torus in \tilde{G} .
 (d) Let T, \tilde{T} be as in (c). Then each root in $\Phi(G, T)$ is the restriction to T of a root in $\Phi(\tilde{G}, \tilde{T})$.
 (e) Let \tilde{T} be as in (c). Then there is some Borel subgroup \tilde{B} of \tilde{G} containing \tilde{T} such that (\tilde{B}, \tilde{T}) is a Borel-torus pair preserved by Γ .

Proof. This is [1, Proposition 3.2]. \square

Let \tilde{T} be a Γ -stable maximal k -torus of \tilde{G} , and let $T = \tilde{T} \cap G = (\tilde{T}^\Gamma)^\circ$. Then we have a norm map $\mathcal{N}_T: \tilde{T} \rightarrow T$ given by

$$\mathcal{N}_T(t) = \prod_{\gamma \in \Gamma} \gamma(t).$$

Suppose G^* (resp. \tilde{G}^*) is a reductive k -group in k -duality with G (resp. \tilde{G}). Let T^* (resp. \tilde{T}^*) be a maximal k -torus of G^* in k -duality with T (resp. \tilde{T}). Let $\delta: X^*(T) \rightarrow X_*(T^*)$ and $\tilde{\delta}: X^*(\tilde{T}) \rightarrow X_*(\tilde{T}^*)$ be duality maps. Then δ and $\tilde{\delta}$ allow one to transfer \mathcal{N}_T to the dual side, giving a k -morphism

$$\hat{\mathcal{N}}_{T^*}: T^* \rightarrow \tilde{T}^*.$$

The main result of [1] is the following.

Theorem 1.2. *There is a unique k -morphism $\hat{\mathcal{N}}_{\tilde{G}, \Gamma}$ from the k -variety of geometric semisimple conjugacy classes in G^* to the analogous variety for \tilde{G}^* with the property that for any maximal k -tori $T^* \subseteq G^*$ and $\tilde{T}^* \subseteq \tilde{G}^*$ as above, and any $s \in T^*$, $\hat{\mathcal{N}}_{T^*}(s) \in \hat{\mathcal{N}}_{\tilde{G}, \Gamma}(x)$, where x is the geometric conjugacy class of s in G^* . Moreover, $\hat{\mathcal{N}}_{\tilde{G}, \Gamma}$ can be refined to give a map $\hat{\mathcal{N}}_{\tilde{G}, \Gamma}^{\text{st}}$ from semisimple stable conjugacy classes in $G^*(k)$ to those in $\tilde{G}^*(k)$.*

2. FACTORING THE CONORM THROUGH LEVI SUBGROUPS

Proposition 2.1. *Let H be a connected reductive k -quasisplit k -group. Let H^* be in k -duality with H . Suppose $T \subseteq H$ is a maximal k -torus, $T^* \subseteq H^*$ is a maximal k -torus in k -duality with T (as in §1) and $L \subseteq H$ is a connected reductive k -subgroup containing T . Let $\Phi^* \subseteq \Phi(H^*, T^*)$ be the root subsystem corresponding to $\Phi := \Phi(L, T) \subseteq \Phi(H, T)$. Suppose that*

Φ^* is closed in $\Phi(H^*, T^*)$. Then H^* has a unique connected reductive k -subgroup L^* containing T^* whose root system is Φ^* . Moreover, if L is a Levi subgroup of G , then L^* is a Levi subgroup of G^* .

Proof. It is clear that the group generated by T^* and the root groups in H^* associated to the roots in Φ^* is a connected reductive group with Φ^* as its root system. Moreover, it is defined over k^{sep} and $\text{Gal}(k)$ -invariant. Thus L^* is defined over k .

Suppose L is a Levi subgroup of G . Then by [5, §3.6], $L^* = C_{G^*}(S^*)$, where S^* is the k -torus $(\bigcap_{\alpha \in \Phi^*} \ker \alpha)^\circ$. Hence L^* is a Levi subgroup of G^* . \square

Remark 2.2. If E/k is an extension over which L^* is quasisplit, then L , as an E -group, is uniquely determined by L^* from [1, §6]. But the k -structure on L can depend upon the choice of T^* .

Proposition 2.3. *Suppose that G lies inside a proper Levi subgroup of \tilde{G} . Then G lies inside a proper, Γ -invariant Levi k -subgroup of \tilde{G} . Suppose that some such Levi k -subgroup \tilde{L} is also k -quasisplit. Let \tilde{L}^* be a Levi k -subgroup of \tilde{G}^* dual to \tilde{L} as in Proposition 2.1. Then $\hat{\mathcal{N}}_{\tilde{G}, \Gamma}$ is a composition of $\hat{\mathcal{N}}_{\tilde{L}, \Gamma}$ with the natural k -morphism i from the k -variety of semisimple geometric conjugacy classes in \tilde{L}^* to the analogous variety for \tilde{G}^* .*

Remark 2.4. Weaker hypotheses suffice. Suppose that G is contained in a group \tilde{L} that is the connected part of the centralizer of a semisimple element of $\tilde{G}(k)$, but is not necessarily a Levi subgroup of \tilde{G} . Under certain circumstances, there is still a corresponding subgroup $\tilde{L}^* \subset \tilde{G}^*$. But we do not pursue the matter here.

Proof. Let \tilde{Z} and Z denote the connected parts of the centers of \tilde{G} and G , respectively, and let $T \subseteq G$ be a maximal k -torus. Let $\tilde{S} = C_{\tilde{G}}(G)^\circ$. Then $\tilde{S} \subseteq C_{\tilde{G}}(T) =: \tilde{T}$, and so \tilde{S} is a k -torus by [2, Cor. 8.4]. Let $\tilde{L} = C_{\tilde{G}}(\tilde{S})$. This is a Levi k -subgroup of \tilde{G} . Any Levi subgroup \tilde{M} of \tilde{G} is the centralizer of some torus in \tilde{G} , and if \tilde{M} contains G , this torus must be contained in \tilde{S} . Thus any Levi subgroup of \tilde{G} containing G must contain \tilde{L} . Our hypothesis thus implies that \tilde{L} is proper. By construction \tilde{L} is Γ -invariant, as desired.

Now suppose that \tilde{L} is a proper, Γ -invariant Levi k -subgroup of \tilde{G} that contains G and that is k -quasisplit. Let T^* be a maximal k -torus of G^* . Let $T \subseteq G$ be a maximal k -torus in k -duality with T^* , let $\tilde{T} = C_{\tilde{G}}(T) \subseteq \tilde{L}$, and let $\tilde{T}^* \subseteq \tilde{G}^*$ be a maximal k -torus in k -duality with \tilde{T} . We can take \tilde{T}^* to lie in \tilde{L}^* . The map $\hat{\mathcal{N}}_{\tilde{G}, \Gamma}$ may be viewed as the map

$$T^*/W(G^*, T^*) \longrightarrow \tilde{T}^*/W(\tilde{G}^*, \tilde{T}^*)$$

induced by the conorm map $\hat{\mathcal{N}}_{T^*}: T^* \longrightarrow \tilde{T}^*$, and $\hat{\mathcal{N}}_{\tilde{L}, \Gamma}$ may be viewed as the map

$$T^*/W(G^*, T^*) \longrightarrow \tilde{T}^*/W(\tilde{L}^*, \tilde{T}^*)$$

induced by $\widehat{\mathcal{N}}_{T^*}$. Since i may be viewed as the natural map

$$T^*/W(\widetilde{L}^*, \widetilde{T}^*) \longrightarrow T^*/W(\widetilde{G}^*, \widetilde{T}^*),$$

it is clear that $\widehat{\mathcal{N}}_{\widetilde{G}, \Gamma} = i \circ \widehat{\mathcal{N}}_{\widetilde{L}, \Gamma}$. \square

Suppose that G is contained in a proper, Γ -invariant Levi k -subgroup \widetilde{L} of \widetilde{G} as in Proposition 2.3, but do not assume that \widetilde{L} is k -quasisplit. Since \widetilde{L} is k^{sep} -quasisplit, \widetilde{L}^* is uniquely determined up to k^{sep} -conjugacy. Working over k^{sep} , one can define a conorm map $\widehat{\mathcal{N}}_{\widetilde{L}, \Gamma}$ from the k^{sep} -variety of semisimple geometric conjugacy classes in G^* to the analogous variety for \widetilde{L}^* . If we choose a maximal k -torus $T \subset G$, then this determines a maximal k -torus $\widetilde{T} \subset \widetilde{L}$, and as in Proposition 2.1, we obtain a k -structure on \widetilde{L}^* , but this structure could depend on a choice of T .

Proposition 2.5. *With the choice of k -structure on \widetilde{L}^* above, the map $\widehat{\mathcal{N}}_{\widetilde{L}, \Gamma}$ is defined over k and has a natural refinement $\widehat{\mathcal{N}}_{\widetilde{L}, \Gamma}^{\text{st}}$ on $G^*(k)$ that takes stable conjugacy classes in $G^*(k)$ to those in $\widetilde{L}^*(k)$. Moreover, $\widehat{\mathcal{N}}_{\widetilde{G}, \Gamma}^{\text{st}}$ is the composition of $\widehat{\mathcal{N}}_{\widetilde{L}, \Gamma}^{\text{st}}$ with the natural map from semisimple stable conjugacy classes in $\widetilde{L}^*(k)$ to those in $\widetilde{G}^*(k)$.*

Remark 2.6. That such a map on stable conjugacy classes from $\widetilde{L}^*(k)$ to $\widetilde{G}^*(k)$ exists follows from the fact that $C_{\widetilde{L}^*}(s)^\circ \subseteq C_{\widetilde{G}^*}(s)^\circ$ for any semisimple element $s \in \widetilde{L}^*(k)$.

Proof. Temporarily replacing k by k^{sep} , Proposition 2.3 shows that $\widehat{\mathcal{N}}_{\widetilde{G}, \Gamma} = i \circ \widehat{\mathcal{N}}_{\widetilde{L}, \Gamma}$. Let $\sigma \in \text{Gal}(k)$. Since $\widehat{\mathcal{N}}_{\widetilde{G}, \Gamma}$ and i are defined over k ,

$$(2.1) \quad i \circ \sigma(\widehat{\mathcal{N}}_{\widetilde{L}, \Gamma}) = \sigma(i \circ \widehat{\mathcal{N}}_{\widetilde{L}, \Gamma}) = \sigma(\widehat{\mathcal{N}}_{\widetilde{G}, \Gamma}) = \widehat{\mathcal{N}}_{\widetilde{G}, \Gamma} = i \circ \widehat{\mathcal{N}}_{\widetilde{L}, \Gamma}.$$

Let X be the irreducible k -variety of semisimple geometric conjugacy classes in G^* , and consider the closed subset

$$Y = \left\{ x \in X \mid \widehat{\mathcal{N}}_{\widetilde{L}, \Gamma}(x) = \sigma(\widehat{\mathcal{N}}_{\widetilde{L}, \Gamma}(x)) \right\}.$$

It follows from (2.1) that Y contains the nonempty open set

$$\left\{ x \in X \mid |i^{-1}(\widehat{\mathcal{N}}_{\widetilde{G}, \Gamma}(x))| = 1 \right\}$$

of X . Since X is irreducible, we must have that $Y = X$, so $\widehat{\mathcal{N}}_{\widetilde{L}, \Gamma} = \sigma(\widehat{\mathcal{N}}_{\widetilde{L}, \Gamma})$, and $\widehat{\mathcal{N}}_{\widetilde{L}, \Gamma}$ is defined over k .

Although \widetilde{L}^* is not necessarily k -quasisplit, the argument in the proof of [1, Thm. 9.1] nevertheless shows that since $\widehat{\mathcal{N}}_{\widetilde{L}, \Gamma}$ is defined over k , it can be refined on $G^*(k)$ to give a map $\widehat{\mathcal{N}}_{\widetilde{L}, \Gamma}^{\text{st}}$ from the semisimple stable conjugacy classes in $G^*(k)$ to those in $\widetilde{L}^*(k)$. The proposition follows. \square

Proposition 2.7. *Let $s \in G^*(k)$. Suppose that $C_{G^*}(s)^\circ$ lies inside a proper Levi subgroup of G^* . Then $C_{G^*}(s)^\circ$ is contained inside a proper Levi k -subgroup L^* of G^* . Moreover, there is a corresponding Levi k -subgroup $\tilde{L} \subseteq \tilde{G}$ on which Γ acts. If $\hat{\mathcal{N}}_{\tilde{L}, \Gamma}$ is a conorm map as in Proposition 2.5 and x is the stable conjugacy class of s in $G^*(k)$, then $\hat{\mathcal{N}}_{\tilde{G}, \Gamma}^{\text{st}}(x)$ is the stable conjugacy class in $\tilde{G}^*(k)$ containing the stable class $\hat{\mathcal{N}}_{\tilde{L}, \Gamma}^{\text{st}}(x)$ in $\tilde{L}^*(k)$.*

As in Remark 2.4, somewhat weaker hypotheses suffice.

Proof. Let Z^* denote the connected part of the center of $C_{G^*}(s)^\circ$. Then Z^* is defined over k . Since $C_{G^*}(s)^\circ \subseteq L^*$, it follows that Z^* is not contained in the center of G^* . Replacing L^* by the centralizer of Z^* if necessary, we may assume that L^* is defined over k .

Let $T^* \subseteq G^*$ be a maximal k -torus in G^* containing s , and let $T \subseteq G$ be a maximal torus in k -duality with T^* . Let L be the proper Levi k -subgroup of G corresponding to L^* , T^* , and T as in Proposition 2.1. Then $L = C_G(S)$ for some k -torus $S \subseteq G$. Let $\tilde{L} = C_{\tilde{G}}(S)$. Then \tilde{L} is a proper, Γ -invariant Levi k -subgroup of \tilde{G} . Let \tilde{T}^* be a maximal torus of \tilde{G}^* in k -duality with \tilde{T} , and let \tilde{L}^* be the proper Levi k -subgroup of \tilde{G}^* corresponding to \tilde{L} , \tilde{T} , and \tilde{T}^* as in Proposition 2.1.

Note that $T^* \subseteq C_{G^*}(s)^\circ \subseteq L^*$ and $\tilde{T}^* \subseteq \tilde{L}^*$. Thus if x is the stable conjugacy class of s in $G(k)$, we obtain that $\hat{\mathcal{N}}_{\tilde{T}}^{\text{st}}(s)$ is contained in both $\hat{\mathcal{N}}^{\text{st}}(x)$ and $\hat{\mathcal{N}}_{\tilde{L}, \Gamma}^{\text{st}}(x)$. \square

3. REDUCTION TO THE CASE WHERE Γ IS SIMPLE

Proposition 3.1. *Suppose $\Gamma_0 \trianglelefteq \Gamma$. Then Γ/Γ_0 acts on $G_0 := (\tilde{G}^{\Gamma_0})^\circ$ via k -automorphisms, and*

$$\hat{\mathcal{N}}_{\tilde{G}, \Gamma} = \hat{\mathcal{N}}_{G_0, \Gamma/\Gamma_0} \circ \hat{\mathcal{N}}_{\tilde{G}, \Gamma_0} \quad \text{and} \quad \hat{\mathcal{N}}_{\tilde{G}, \Gamma}^{\text{st}} = \hat{\mathcal{N}}_{G_0, \Gamma/\Gamma_0}^{\text{st}} \circ \hat{\mathcal{N}}_{\tilde{G}, \Gamma_0}^{\text{st}}.$$

Proof. Let $\gamma \in \Gamma$ be a representative for an element $\gamma\Gamma_0 \in \Gamma/\Gamma_0$. Then γ preserves G_0 . Moreover all elements of $\gamma\Gamma_0$ act in the same way on G_0 . Thus, we have an action of Γ/Γ_0 on G_0 , and $(G_0)^{\Gamma/\Gamma_0} = (\tilde{G}^\Gamma)^\circ = G$.

Let $T^* \subseteq G^*$ be a maximal k -torus. Choose maximal a k -torus $T \subseteq G$, as in [1, Prop. 6.4]. Let $\tilde{T} = C_{\tilde{G}}(T)$ and $T_0 = C_{G_0}(T)$. By Proposition 1.1(c), these are maximal k -tori in \tilde{G} and G_0 . Choose maximal k -tori $T_0^* \subseteq G_0^*$, and $\tilde{T}^* \subseteq \tilde{G}^*$ as in [1, Prop. 6.4]. Let

$$\mathcal{N}_{T, \Gamma}: \tilde{T} \longrightarrow T, \quad \mathcal{N}_{T_0, \Gamma_0}: \tilde{T} \longrightarrow T_0, \quad \mathcal{N}_{T, \Gamma/\Gamma_0}: T_0 \longrightarrow T$$

denote the norm maps corresponding to the actions of Γ , Γ_0 , and Γ/Γ_0 on \tilde{T} , \tilde{T} (again), and T_0 , respectively. Let

$$\hat{\mathcal{N}}_{T^*, \Gamma}: T^* \longrightarrow \tilde{T}^*, \quad \hat{\mathcal{N}}_{T_0^*, \Gamma_0}: T_0^* \longrightarrow \tilde{T}^*, \quad \hat{\mathcal{N}}_{T^*, \Gamma/\Gamma_0}: T^* \longrightarrow T_0^*$$

denote the corresponding conorm maps. Since $\mathcal{N}_{T,\Gamma} = \mathcal{N}_{T,\Gamma/\Gamma_0} \circ \mathcal{N}_{T_0\Gamma_0}$, we have that $\hat{\mathcal{N}}_{T^*,\Gamma} = \hat{\mathcal{N}}_{T_0^*,\Gamma_0} \circ \hat{\mathcal{N}}_{T^*,\Gamma/\Gamma_0}$. Since the maps $\hat{\mathcal{N}}_{\tilde{G},\Gamma}$, $\hat{\mathcal{N}}_{G_0,\Gamma/\Gamma_0}$, and $\hat{\mathcal{N}}_{\tilde{G},\Gamma_0}$ on semisimple conjugacy classes (and $\hat{\mathcal{N}}_{\tilde{G},\Gamma}^{\text{st}}$, $\hat{\mathcal{N}}_{G_0,\Gamma/\Gamma_0}^{\text{st}}$, and $\hat{\mathcal{N}}_{\tilde{G},\Gamma_0}^{\text{st}}$ on stable classes) are compatible with the respective maps $\hat{\mathcal{N}}_{T^*,\Gamma}$, $\hat{\mathcal{N}}_{T^*,\Gamma/\Gamma_0}$, and $\hat{\mathcal{N}}_{T_0^*,\Gamma_0}$ on maximal k -tori, our result follows. \square

Thus, where convenient, we may assume that Γ is simple.

4. THE CASE WHERE Γ PERMUTES A PRODUCT

Definition 4.1. A *torus-independent conorm function* for G^* is an algebraic morphism $\hat{N}: G^* \rightarrow \tilde{G}^*$ (not necessarily a group homomorphism) whose restriction to each maximal k -torus $T^* \subseteq G^*$ equals $\hat{\mathcal{N}}_{T^*}$ (defined with respect to some choices of T , δ , \tilde{T}^* , and $\tilde{\delta}$).

Proposition 4.2. Let \tilde{G} be a direct product $\prod H$ of r copies of a connected reductive k^{sep} -group H . Let Γ be a transitive permutation group of the factors of \tilde{G} . Let $\text{diag}: H \rightarrow \prod H$ denote the diagonal embedding. Then the fixed-point group $G = \tilde{G}^\Gamma$ is $\text{diag}(H)$, and we can take $\tilde{G}^* = \prod H^*$, and $G^* = \text{diag}(H^*)$. Then we have a torus-independent conorm map $\hat{N}: G^* \rightarrow \tilde{G}^*$ given by $\text{diag}(x) \mapsto \text{diag}(x^m)$, where m is the order of the stabilizer in Γ of one (hence any) factor of \tilde{G} .

Proof. Let \dot{T} be any maximal torus of H . Then $T := \text{diag}(\dot{T})$ (resp. $\tilde{T} := \prod \dot{T}$) is a maximal torus of G (resp. \tilde{G}). Moreover, $\tilde{T} = C_{\tilde{G}}(T)$.

Note that we can take the dual group \tilde{G}^* to be $\prod H^*$, since the latter is in k -duality with \tilde{G} , and that we may choose $\tilde{T}^* = \prod \dot{T}^*$, where \dot{T}^* is a torus in H^* dual to \dot{T} . Note also that

$$\begin{aligned} \mathbf{X}^*(\tilde{T}) &= \prod \mathbf{X}^*(\dot{T}), & \mathbf{X}^*(T) &= \text{diag}(\mathbf{X}^*(\dot{T})) \subseteq \mathbf{X}^*(\tilde{T}), \\ \mathbf{X}_*(\tilde{T}^*) &= \prod \mathbf{X}_*(\dot{T}^*), & \mathbf{X}_*(T^*) &= \text{diag}(\mathbf{X}_*(\dot{T}^*)) \subseteq \mathbf{X}_*(\tilde{T}^*), \end{aligned}$$

where here diag denotes the obvious diagonal embeddings of lattices. Moreover, we may choose the isomorphism $\tilde{\delta}: \prod \mathbf{X}^*(T) \rightarrow \prod \mathbf{X}_*(T^*)$ so that it respects the product structure of each lattice, and so that its components are all equal to a fixed isomorphism $\delta: \mathbf{X}^*(\dot{T}) \rightarrow \mathbf{X}_*(\dot{T}^*)$, and we may set $\delta: \mathbf{X}^*(T) \rightarrow \mathbf{X}_*(T^*)$ equal to $\text{diag}(\delta)$.

The norm map $\mathcal{N}_T: \tilde{T} \rightarrow T$ sends an r -tuple $(t_i) \in \tilde{T}$ to $\text{diag}(\prod t_i^m)$. We may compute the corresponding map $\mathcal{N}_T^*: \mathbf{X}^*(T) \rightarrow \mathbf{X}^*(\tilde{T})$ as follows. If $\chi \in \mathbf{X}^*(T)$, then $\chi = \text{diag}(\dot{\chi})$ for some $\dot{\chi} \in \mathbf{X}^*(\dot{T})$. For $(t_i) \in \tilde{T}$, we have

$$(\mathcal{N}_T^* \chi)((t_i)) = \chi(\mathcal{N}_T((t_i))) = \prod \dot{\chi}^m(t_i) = (\text{diag}(\dot{\chi}^m))((t_i)).$$

Thus $\mathcal{N}_T^*(\text{diag}(\dot{\chi})) = \text{diag}(\dot{\chi}^m)$. It follows that the conorm $\hat{\mathcal{N}}_{T^*}: T^* \rightarrow \tilde{T}^*$ is given by $\hat{\mathcal{N}}_{T^*}(\text{diag}(x)) = \text{diag}(x^m)$ for $x \in T^*$, where diag now represents

the natural diagonal embedding $H^* \longrightarrow \tilde{G}^*$. Thus $\hat{N} : \text{diag}(x) \mapsto \text{diag}(x^m)$ defines a torus-independent conorm map $G^* \longrightarrow \tilde{G}^*$. \square

Corollary 4.3. *In the situation described in Proposition 4.2, suppose that Γ acts simply and transitively. Then the diagonal embedding of G^* in \tilde{G}^* is a torus-independent conorm function.* \square

Corollary 4.4. *Suppose that the action of Γ on \tilde{G} is trivial. Then we have a torus-independent conorm function for G^* given by $s \mapsto s^{|\Gamma|}$.* \square

5. COMPATIBILITY WITH ISOGENIES

Suppose that $\tilde{\phi} : \tilde{G} \longrightarrow \tilde{G}'$ is a k -isogeny with Γ -invariant kernel. Then the action of Γ on \tilde{G} naturally gives rise to an action of Γ on \tilde{G}' via k -automorphisms that all preserve a Borel-torus pair. Let $G' = (\tilde{G}'^\Gamma)^\circ$. Let \tilde{G}'^* be a connected reductive k -group in k -duality with \tilde{G}' . Then $\tilde{\phi}$ determines a finite k -subgroup \tilde{Z}'^* of \tilde{G}'^* and hence a k isogeny $\tilde{\phi}^\wedge : \tilde{G}'^* \longrightarrow \tilde{G}^*$, where $\tilde{G}^* := \tilde{G}'^* / \tilde{Z}'^*$ is in k -duality with \tilde{G} . Similarly, we obtain groups G^* and G'^* in k -duality with G and G' , respectively, and a k -isogeny $\phi^\wedge : G^* \longrightarrow G'^*$.

Let \tilde{T} be a Γ -stable maximal k -torus of \tilde{G} , and let $\tilde{T}' = \tilde{\phi}(\tilde{T})$. Then Γ acts on \tilde{T}' . Taking connected parts of groups of Γ -fixed points, we obtain a k -isogeny of tori $\phi : T \longrightarrow T'$, and norm maps $\mathcal{N} := \mathcal{N}_T$ and $\mathcal{N}' := \mathcal{N}_{T'}$ (defined over k) that make the following square commute:

$$(5.1) \quad \begin{array}{ccc} \tilde{T} & \xrightarrow{\tilde{\phi}} & \tilde{T}' \\ \mathcal{N} \downarrow & & \downarrow \mathcal{N}' \\ T & \xrightarrow{\phi} & T' \end{array}$$

Suppose that $\tilde{T}^* \subseteq \tilde{G}^*$ and $T^* \subset G^*$ are maximal tori in k -duality with \tilde{T} and T , respectively, via the duality maps

$$\delta_{\tilde{T}} : \mathbf{X}^*(\tilde{T}) \longrightarrow \mathbf{X}_*(\tilde{T}^*) \quad \text{and} \quad \delta_T : \mathbf{X}^*(T) \longrightarrow \mathbf{X}_*(T^*),$$

and that $\tilde{T}'^* \subseteq \tilde{G}'^*$ and $T'^* \subseteq G'^*$ are maximal k -tori in k -duality with \tilde{T}' and T' , respectively, via the duality maps

$$\delta_{\tilde{T}'} : \mathbf{X}^*(\tilde{T}') \longrightarrow \mathbf{X}_*(\tilde{T}'^*) \quad \text{and} \quad \delta_{T'} : \mathbf{X}^*(T') \longrightarrow \mathbf{X}_*(T'^*).$$

These choices of duality maps determine k -homomorphisms

$$\hat{\mathcal{N}} := \hat{\mathcal{N}}_{T^*} : T^* \longrightarrow \tilde{T}^* \quad \text{and} \quad \hat{\mathcal{N}}' := \hat{\mathcal{N}}_{T'^*} : T'^* \longrightarrow \tilde{T}'^*.$$

Proposition 5.1. *The above maps of tori satisfy $\tilde{\phi}^\wedge \circ \hat{\mathcal{N}}' = \hat{\mathcal{N}} \circ \phi^\wedge$.*

Proof. Consider the following cube, all of whose horizontal edges represent isomorphisms:

$$\begin{array}{ccccc}
 & & \mathbf{X}^*(\tilde{T}) & \xrightarrow{\delta_{\tilde{T}}} & \mathbf{X}_*(\tilde{T}^*) \\
 & \nearrow \tilde{\phi}^* & \uparrow & & \nearrow \tilde{\phi}_*^\wedge \\
 \mathbf{X}^*(\tilde{T}') & \xrightarrow{\delta_{\tilde{T}'}} & \mathbf{X}_*(\tilde{T}'^*) & & \mathbf{X}_*(\tilde{T}^*) \\
 \uparrow \mathcal{N}^* & & \uparrow \mathcal{N}'_* & & \uparrow \hat{\mathcal{N}}_* \\
 & \nearrow \phi^* & \mathbf{X}^*(T) & \xrightarrow{\delta_T} & \mathbf{X}_*(T^*) \\
 & & \uparrow & & \nearrow \phi_*^\wedge \\
 \mathbf{X}^*(T') & \xrightarrow{\delta_{T'}} & \mathbf{X}_*(T'^*) & & \mathbf{X}_*(T^*)
 \end{array}$$

The front and back faces commute by the definitions of $\hat{\mathcal{N}}'_*$ and $\hat{\mathcal{N}}_*$. The left-hand face commutes from applying the functor \mathbf{X}^* to (5.1). The top and bottom faces commute because of the definitions of $\tilde{\phi}_*^\wedge$ and ϕ_*^\wedge . Therefore, the right-hand face commutes, and thus also the square

$$\begin{array}{ccc}
 \tilde{T}'^* & \xrightarrow{\tilde{\phi}^\wedge} & \tilde{T}^* \\
 \hat{\mathcal{N}}' \uparrow & & \uparrow \hat{\mathcal{N}} \\
 T'^* & \xrightarrow{\phi^\wedge} & T^*
 \end{array}$$

□

Note that ϕ^\wedge (resp. $\tilde{\phi}^\wedge$) naturally determines a k -morphism, which we will also denote by ϕ^\wedge (resp. $\tilde{\phi}^\wedge$) from the k -variety of geometric conjugacy classes in G'^* (resp. \tilde{G}'^*) to the analogous variety for G^* (resp. \tilde{G}^*). Also note that for all semisimple $s \in G'^*(k)$ and $t \in \tilde{G}'^*(k)$,

$$\phi^\wedge(C_{G'^*}(s)^\circ) = C_{G^*}(\phi^\wedge(s))^\circ \quad \text{and} \quad \tilde{\phi}^\wedge(C_{\tilde{G}'^*}(t)^\circ) = C_{\tilde{G}^*}(\tilde{\phi}^\wedge(t))^\circ.$$

Thus the maps ϕ^\wedge and $\tilde{\phi}^\wedge$ on geometric conjugacy classes just defined can be refined to give corresponding maps on stable conjugacy classes.

The following corollary now follows immediately from Proposition 5.1.

Corollary 5.2. *In the above notation, we have*

$$\tilde{\phi}^\wedge \circ \hat{\mathcal{N}}_{\tilde{G}', \Gamma} = \hat{\mathcal{N}}_{\tilde{G}, \Gamma} \circ \phi^\wedge \quad \text{and} \quad \tilde{\phi}^\wedge \circ \hat{\mathcal{N}}_{\tilde{G}', \Gamma}^{\text{st}} = \hat{\mathcal{N}}_{\tilde{G}, \Gamma}^{\text{st}} \circ \phi^\wedge.$$

6. PRODUCT DECOMPOSITIONS

From Corollary 5.2, we may as well assume that \tilde{G} is a direct product of a k -torus and a collection of almost k -simple factors. Moreover, Γ clearly permutes these almost k -simple factors and stabilizes the torus. Since the conorm on a torus is an explicitly defined homomorphism, we can reduce

the problem of understanding conorms explicitly to the case where \tilde{G} is a direct product $\prod \tilde{M}$ of a collection of almost k -simple groups, and Γ acts transitively on the factors. In general, the action of Γ could be very complicated, but we consider two subcases: Γ acts by permuting the coordinates of elements of $\tilde{G} = \prod \tilde{M}$, or \tilde{G} is itself k -simple.

If Γ permutes the coordinate of the factors, then Proposition 4.2 applies.

On the other hand, if \tilde{G} is k -simple, then from Corollary 5.2 (again), we may assume that \tilde{G} is obtained via restriction of scalars from an absolutely simple E group \tilde{H} , where E/k is some finite separable extension. In particular, \tilde{G} is k^{sep} -isomorphic to a direct product of absolutely almost simple factors, and $\text{Gal}(k)$ acts transitively on the set of factors. Moreover, Γ permutes these factors. By Proposition 3.1, we may assume that Γ is simple. It follows from an argument in the proof of [1, Prop. 3.2] that Γ either acts simply on the set of factors or preserves each factor, acting in the same way on each.

In the former case, using elements of Γ to identify factors, one finds that Γ acts on the direct product \tilde{G} by permuting coordinates corresponding to the absolutely simple factors. Hence the conorm arises from a torus-independent conorm function as in Proposition 4.2.

In the latter case, the elements of Γ induce E -automorphisms of \tilde{H} which preserve a common Borel-torus pair, and the problem of understanding the k -morphism $\hat{\mathcal{N}}_{\tilde{G}, \Gamma}$ explicitly reduces to understanding the simpler E -morphism $\hat{\mathcal{N}}_{\tilde{H}, \Gamma}$. It is not hard to see that the stable conjugacy classes in $\tilde{G}(k)$ and $\tilde{H}(E)$ coincide, so the problem of understanding the map $\hat{\mathcal{N}}_{\tilde{G}, \Gamma}^{\text{st}}$ explicitly similarly reduces to understanding $\hat{\mathcal{N}}_{\tilde{H}, \Gamma}^{\text{st}}$.

7. REDUCTION TO THE FIXED-PINNING CASE

We now address the relationship between the conorm map corresponding to the given action of the group Γ and that corresponding to another action that fixes a pinning.

Let $\varphi: \Gamma \rightarrow \text{Aut}_k(\tilde{G})$ be the homomorphism that describes how Γ acts on \tilde{G} .

Recall that Γ fixes a Borel-torus pair $(\tilde{B}_0, \tilde{T}_0)$. By [1, Remark 3.3(iii)], we may furthermore assume that \tilde{T}_0 is defined over k . Let $\tilde{\Delta} \subseteq \tilde{\Phi} := \Phi(\tilde{G}, \tilde{T}_0)$ denote the corresponding set of simple roots. A *pinning* for the triple $(\tilde{G}, \tilde{B}_0, \tilde{T}_0)$ is a set $\{X_\alpha \mid \alpha \in \tilde{\Delta}\}$, where each X_α is a non-zero vector in the α -root space $\text{Lie}(\tilde{G})_\alpha$ of $\text{Lie}(\tilde{G})$.

Proposition 7.1. *There is a homomorphism $\varphi: \Gamma \rightarrow \text{Aut}_k(\tilde{G})$ such that for all $\gamma \in \Gamma$, $\varphi(\gamma)$ preserves $(\tilde{B}_0, \tilde{T}_0)$, acts on \tilde{T}_0 in the same way as $\varphi(\gamma)$, and fixes a pinning for $(\tilde{G}, \tilde{B}_0, \tilde{T}_0)$.*

Proof. For each $\text{Gal}(k)$ -orbit of roots in $\tilde{\Phi}$, choose a representative root α . Let E_α denote the fixed field of the stabilizer of α in $\text{Gal}(k)$. Choose a nonzero root vector $X_\alpha \in \text{Lie}(\tilde{G})(E_\alpha)$. If $\beta \in \tilde{\Phi}$ and $\beta = \sigma(\alpha)$ for some $\sigma \in \text{Gal}(k)$, then let $X_\beta = \sigma(X_\alpha)$. It is easy to see that this is independent of the choice of σ . We thus obtain a set of root vectors $\{X_\alpha \mid \alpha \in \tilde{\Phi}\}$ that is permuted by $\text{Gal}(k)$.

Consider the split short exact sequence

$$1 \longrightarrow \text{Imm}(\tilde{G}) \longrightarrow \text{Aut}(\tilde{G}) \xrightarrow[\psi]{\pi} \text{Aut}(\Delta(\tilde{B}_0, \tilde{T}_0)) \longrightarrow 1,$$

where $\text{Aut}(\Delta(\tilde{B}_0, \tilde{T}_0))$ is the group of automorphisms of the based root datum associated to $(\tilde{B}_0, \tilde{T}_0)$ and ψ is the splitting determined by our choice of pinning. Define a homomorphism $\underline{\varphi}: \Gamma \longrightarrow \text{Aut}(\tilde{G})$ by $\underline{\varphi} = \psi \circ \pi \circ \varphi$. Thus, for each $\gamma \in \Gamma$, we have that $\underline{\varphi}(\gamma)$ fixes our pinning, and there is some $t_\gamma \in T(k^{\text{sep}})$ such that $\underline{\varphi}(\gamma)\varphi(\gamma)^{-1} = \text{Int}(t_\gamma)$.

To show that $\underline{\varphi}(\gamma)$ is defined over k it suffices to show that $\text{Int}(t_\gamma)$ is defined over k . Since $\text{Int}(t_\gamma)$ is defined over k^{sep} , it is enough to verify that $\text{Int}(t_\gamma)$ is fixed by $\text{Gal}(k)$. But if $\alpha \in \tilde{\Delta}$ and $\sigma \in \text{Gal}(k)$, we have

$$\begin{aligned} \text{Ad}(\sigma(t_\gamma))^{-1}(X_\alpha) &= (\sigma(\text{Ad}(t_\gamma))^{-1})(X_\alpha) \\ &= \sigma((\varphi(\gamma)\underline{\varphi}(\gamma)^{-1})(\sigma^{-1}X_\alpha)) \\ &= \sigma((\varphi(\gamma))(X_{\gamma^{-1}(\sigma^{-1}(\alpha))})) \\ &= \sigma((\varphi(\gamma))(X_{\sigma^{-1}(\gamma^{-1}(\alpha))})) \\ &= \sigma((\varphi(\gamma))(\sigma^{-1}X_{\gamma^{-1}(\alpha)})) \\ &= (\sigma(\varphi(\gamma)))(X_{\gamma^{-1}(\alpha)}) \\ &= (\varphi(\gamma))(X_{\gamma^{-1}(\alpha)}) \\ &= (\varphi(\gamma)\underline{\varphi}(\gamma)^{-1})(X_\alpha) \\ &= \text{Ad}(t_\gamma)^{-1}(X_\alpha). \end{aligned}$$

It follows that every root in $\tilde{\Delta}$ takes the same values on t_γ and $\sigma(t_\gamma)$. Hence $t_\gamma^{-1}\sigma(t_\gamma)$ is central, and $\text{Int}(t_\gamma)$ must be fixed by $\text{Gal}(k)$. \square

Remark 7.2. In general, it is easily seen that $\Phi(G, T)$ consists of all restrictions $i^*\alpha$ of roots $\alpha \in \tilde{\Phi}$ such that

$$\sum_{\gamma \in \Gamma} \varphi(\gamma) \cdot X_\alpha \neq 0.$$

This condition is equivalent to the condition that $\varphi(\gamma)$ act trivially on $\text{Lie}(\tilde{G})_\alpha$ for each $\gamma \in \Gamma_\alpha := \text{stab}_\Gamma(\alpha)$.

Choose maximal k -tori $\tilde{T}_0^* \subseteq \tilde{G}_0^*$ and $T_0^* \subseteq G_0^*$ in k -duality with \tilde{T}_0 and T_0 , respectively. Let

$$\tilde{\Phi} = \Phi(\tilde{G}, \tilde{T}_0), \quad \tilde{\Phi}^* = \Phi(\tilde{G}^*, \tilde{T}_0^*), \quad \Phi = \Phi(G, T_0), \quad \Phi^* = \Phi(G^*, T_0^*).$$

Also, let $\underline{G} = (\tilde{G}^{\varphi(\Gamma)})^\circ$ and let \underline{G}^* be a k -group that contains T_0^* and is in k -duality with \underline{G} . Define

$$\underline{\Phi} = \Phi(\underline{G}, T_0), \quad \underline{\Phi}^* = \Phi(\underline{G}^*, T_0^*).$$

We can identify Φ and $\underline{\Phi}$ with subsets of $V^*(T_0) = V^*(\tilde{T}_0)^\Gamma$ as in [1, §2]. Recall that the inclusion map $i: T_0 \longrightarrow \tilde{T}_0$ induces the restriction map

$$i^*: V^*(\tilde{T}_0) \longrightarrow V^*(T_0) = V^*(\tilde{T}_0)^\Gamma,$$

given by $i^*(\alpha) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \gamma(\alpha)$. We observe that $W(G, T_0)$ embeds naturally in $W(\tilde{G}, \tilde{T}_0)^{\varphi(\Gamma)}$, which can be identified canonically with $W(\underline{G}, T_0)$ (see [4, §1.1]).

Lemma 7.3. *Suppose that the stabilizer in $\varphi(\Gamma)$ of every irreducible subsystem of $\tilde{\Phi}$ of type A_{2n} acts either trivially or faithfully on that subsystem. Then, as subsets of $V^*(\tilde{T}_0)$, we have the inclusion $\Phi \subseteq \underline{\Phi}$.*

Proof. We may assume that $\tilde{\Phi}$ is a single Γ -orbit of irreducible subsystems. For each $\alpha \in \tilde{\Phi}$, let Γ_α denote the stabilizer of α in Γ .

For each $\alpha \in \tilde{\Delta}$, we have that $\varphi(\Gamma_\alpha)$ acts trivially on $\text{Lie}(G)_\alpha$. If the irreducible subsystems of $\tilde{\Phi}$ are not of type A_{2n} , it follows from [4, §1.3.3] and Remark 7.2 that $\varphi(\Gamma_\alpha)$ acts trivially on $\text{Lie}(\tilde{G})_\alpha$ for every $\alpha \in \tilde{\Phi}$. Thus $\Phi \subseteq \underline{\Phi}$.

Now suppose that $\tilde{\Phi}$ is a Γ -orbit of irreducible subsystems of type A_{2n} . If the stabilizer of some irreducible subsystem Ψ in $\varphi(\Gamma)$ acts trivially on Ψ , then the stabilizer of Ψ in $\varphi(\Gamma)$ does so as well and, moreover, acts trivially on $\text{Lie}(\tilde{G})_\alpha$ for every $\alpha \in \Psi$. Thus $\varphi(\Gamma_\alpha)$ acts trivially on $\text{Lie}(\tilde{G})_\alpha$ for each $\alpha \in \Psi$. Since Γ acts transitively on the irreducible subsystems of $\tilde{\Phi}$, the preceding statement holds for all $\alpha \in \tilde{\Phi}$. Thus $\Phi \subseteq \underline{\Phi}$.

On the other hand, suppose that $\varphi(\Gamma)$ acts faithfully on some irreducible subsystem Ψ of $\tilde{\Phi}$ (still assumed to be of type A_{2n}). Let $\Gamma_\Psi \subseteq \Gamma$ be the stabilizer of Ψ . Then $\varphi(\Gamma_\Psi)$ must have order 2, and $\varphi(\Gamma_\alpha)$ must be trivial for every $\alpha \in \Psi \cap \tilde{\Delta}$. Since Γ acts transitively on the irreducible subsystems of $\tilde{\Phi}$, $\varphi(\Gamma_\alpha)$ must therefore be trivial for every $\alpha \in \tilde{\Delta}$. Thus, $\varphi = \underline{\varphi}$, so $\Phi = \underline{\Phi}$. \square

Lemma 7.4. *Suppose that the stabilizer in $\varphi(\Gamma)$ of every irreducible subsystem of $\tilde{\Phi}$ is cyclic and acts faithfully on that subsystem. Then, as subsets of $V^*(\tilde{T}_0)$, we have the inclusion $\underline{\Phi}_{\text{short}} \subset \Phi$.*

Remark 7.5. The assumption that the stabilizer be cyclic rules out only the case in which $\tilde{\Phi}$ has an irreducible subsystem of type D_4 , whose stabilizer

acts on the subsystem via the full symmetric group S_3 . In this situation, the lemma can fail.

Proof. We may assume that Γ acts transitively on the set of irreducible subsystems of $\tilde{\Phi}$. We observe that by Remark 7.2, it suffices to show that if $\alpha \in \tilde{\Phi}$ restricts to a short root in $\underline{\Phi}$, then the stabilizer in $\varphi(\Gamma)$ of α is trivial.

If the stabilizer in $\varphi(\Gamma)$ of some (hence every) irreducible subsystem Ψ of $\tilde{\Phi}$ acts trivially on Ψ (for example, when Ψ is not simply laced), then, by the assumption of faithfulness, this stabilizer must therefore be trivial. It follows that every root in $\tilde{\Phi}$ has trivial stabilizer in $\varphi(\Gamma)$. Thus by Remark 7.2, $\Phi = \underline{\Phi}$, and the lemma follows in this case.

Now suppose that the stabilizer Γ_Ψ in $\varphi(\Gamma)$ of some (hence every) irreducible $\Psi \subset \tilde{\Phi}$ acts nontrivially on Ψ . Then Ψ must be simply laced and must contain a root α that is fixed by Γ_Ψ and a root β that is not fixed by Γ_Ψ . Moreover, since Γ_Ψ acts faithfully on Ψ , it follows that the stabilizer of β in $\varphi(\Gamma)$ is trivial.

If Ψ is of type A_{2n} , then no simple root in $\tilde{\Delta}$ is fixed by $\varphi(\Gamma)$. Thus every simple root has trivial stabilizer, and Remark 7.2 implies that $\varphi = \underline{\varphi}$. Therefore, we may assume that Ψ is not of type A_{2n} .

We now consider the lengths of α and β . Let (\cdot, \cdot) denote a $W(\tilde{G}, \tilde{T}_0) \rtimes \Gamma$ -invariant inner product on $V^*(\tilde{T}_0)$. We have

$$\begin{aligned}
 (i^*\beta, i^*\beta) &= \frac{1}{|\Gamma|^2} \sum_{\gamma, \gamma' \in \Gamma} (\gamma(\beta), \gamma'(\beta)) \\
 &= \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} (\gamma(\beta), \beta) \\
 (7.1) \qquad &= \frac{1}{|\Gamma \cdot \beta|} \sum_{\beta' \in \Gamma \cdot \beta} (\beta', \beta).
 \end{aligned}$$

By [4, §1.3.5], since Ψ is not of type A_{2n} , β is orthogonal to every other root in its Γ -orbit. Thus (7.1) is equal to

$$\frac{1}{|\Gamma \cdot \beta|} (\beta, \beta) < (\beta, \beta) = (\alpha, \alpha) = (i^*\alpha, i^*\alpha).$$

Hence $\underline{\Phi}$ contains roots of different lengths, and moreover, a root in $\underline{\Phi}$ is short if and only if it is the restriction of a root in $\tilde{\Phi}$ whose stabilizer in $\varphi(\Gamma)$ is trivial. The lemma then follows from the observation at the beginning of the proof. \square

Corollary 7.6. *The roots in $\underline{\Phi}^*$ are not all of the same length, and if $\underline{\Phi}_{\text{long}}^*$ denotes the set of long roots in $\underline{\Phi}^*$, then we have $\underline{\Phi}_{\text{long}}^* \subseteq \Phi^* \subseteq \underline{\Phi}^*$.*

Proposition 7.7. *Suppose that the stabilizer in $\varphi(\Gamma)$ of every irreducible subsystem of $\tilde{\Phi}$ is cyclic and acts faithfully on that subsystem.*

- (a) *There is a unique embedding of k -groups $G^* \hookrightarrow \underline{G}^*$ that restricts to the identity on T_0^* and is compatible with the inclusion of root systems $\Phi^* \hookrightarrow \underline{\Phi}^*$ from Corollary 7.6.*
- (b) *The conorm map on geometric conjugacy classes factors through this embedding: $\widehat{\mathcal{N}}_{\tilde{G}} = \widehat{\mathcal{N}}_{\tilde{G}} \circ i$, where i is the natural k -morphism from the variety of semisimple geometric conjugacy classes in G^* to that for \underline{G}^* induced by the embedding in (a), and $\widehat{\mathcal{N}}_{\tilde{G}}$ (resp. $\widehat{\mathcal{N}}_{\tilde{G}}$) is the conorm map induced by the action of Γ on \tilde{G} via φ (resp. $\underline{\varphi}$).*
- (c) *Let $\widehat{\mathcal{N}}_{\tilde{G}}^{\text{st}}$ (resp. $\widehat{\mathcal{N}}_{\tilde{G}}^{\text{st}}$) be the refinement of $\widehat{\mathcal{N}}_{\tilde{G}}$ to the set of stable conjugacy classes in $G^*(k)$ (resp. $\underline{G}^*(k)$). The conorm map on stable conjugacy classes satisfies $\widehat{\mathcal{N}}_{\tilde{G}}^{\text{st}} = \widehat{\mathcal{N}}_{\tilde{G}}^{\text{st}} \circ j$, where j is the natural map from semisimple stable conjugacy classes in $G^*(k)$ to those in $\underline{G}^*(k)$ induced by the embedding in (a).*

Proof. Since G^* and \underline{G}^* share the torus T_0^* , to establish the embedding in (a), it will be enough to show that Φ^* embeds as a closed subsystem of $\underline{\Phi}^*$. By Corollary 7.6, we have $\Phi_{\text{long}}^* \subseteq \Phi^* \subseteq \underline{\Phi}^*$.

Consider two roots $\alpha_1, \alpha_2 \in \Phi^*$. Suppose that $\alpha_1 + \alpha_2 \in \underline{\Phi}^*$. We must show that $\alpha_1 + \alpha_2 \in \Phi^*$. Let $\Psi = \Phi^* \cap \text{span}\{\alpha_1, \alpha_2\}$, and $\underline{\Psi} = \underline{\Phi}^* \cap \text{span}\{\alpha_1, \alpha_2\}$. Then $\Psi \subseteq \underline{\Psi}$ are two-dimensional root systems whose long roots coincide. Note that $\alpha_1, \alpha_2 \in \Psi$ and $\alpha_1 + \alpha_2 \in \underline{\Psi}$. A quick examination all two-dimensional root systems shows that $\alpha_1 + \alpha_2 \in \Psi$. Part (a) follows.

To prove (b), note that $\widehat{\mathcal{N}}_{\tilde{G}}$ may be viewed as the map

$$T_0^*/W(G^*, T_0^*) \longrightarrow \tilde{T}_0^*/W(\tilde{G}^*, \tilde{T}_0^*)$$

induced by the conorm map $\widehat{\mathcal{N}}_{T_0^*}: T_0^* \longrightarrow \tilde{T}_0^*$, and that $\widehat{\mathcal{N}}_{\tilde{G}}$ may be viewed as the map

$$T_0^*/W(\underline{G}^*, T_0^*) \longrightarrow \tilde{T}_0^*/W(\tilde{G}^*, \tilde{T}_0^*)$$

induced by $\widehat{\mathcal{N}}_{T_0^*}$. Since i may be viewed as the natural map

$$T_0^*/W(G^*, T_0^*) \longrightarrow T_0^*/W(\underline{G}^*, T_0^*),$$

it is clear that $\widehat{\mathcal{N}}_{\tilde{G}} = \widehat{\mathcal{N}}_{\tilde{G}} \circ i$.

It remains to prove (c). Let s be a semisimple element of $G^*(k)$ and let T^* be a maximal k -torus containing s . Pick maximal k -tori $T \subseteq G$, $\tilde{T} \subseteq \tilde{G}$, and $\tilde{T}^* \subseteq \tilde{G}^*$ as in [1, §7].

Since T^* is also maximal in \underline{G}^* , we can similarly pick maximal k -tori $\underline{T} \subseteq \underline{G}$, $\underline{\tilde{T}} \subseteq \underline{\tilde{G}}$, and $\underline{\tilde{T}}^* \subseteq \underline{\tilde{G}^*}$. Write $T = {}^g T_0$, $\underline{T} = {}^{\underline{g}} T_0$, and $T^* = {}^{g^*} T_0^*$, with $g \in G(k^{\text{sep}})$, $\underline{g} \in \underline{G}(k^{\text{sep}})$, and $g^* \in G^*(k^{\text{sep}})$. As in [1, Remark 6.6], we may choose g, \underline{g}, g^* such that for all $\sigma \in \text{Gal}(k)$, the images of $g^{-1}\sigma(g)$ in $W(G, T_0)$, $\underline{g}^{-1}\sigma(\underline{g})$ in $W(\underline{G}, T_0)$, and $g^{*-1}\sigma(g^*)$ in $W(G^*, T_0^*)$, are equal under the identification and embedding $W(G^*, T_0^*) = W(G, T_0) \subseteq W(\underline{G}, T_0)$.

Consider $h = \underline{g}g^{-1} \in \tilde{G}(k^{\text{sep}})$. For all $\sigma \in \text{Gal}(k)$, we have

$$\sigma(h) = \sigma(\underline{g})\sigma(g)^{-1} = \underline{g}(\underline{g}^{-1}\sigma(\underline{g}))\sigma(g)^{-1} \equiv \underline{g}(g^{-1}\sigma(g))\sigma(g)^{-1} = h,$$

where the congruence indicates equality of the images in $W(\underline{G}, T_0)$. Thus, $\text{Int}(h)$ is defined over k as an isomorphism $T \rightarrow \underline{T}$. It also defines a k -isomorphism $\tilde{T} \rightarrow \tilde{\underline{T}}$. Therefore, since $\tilde{T}^* \subseteq \tilde{G}^*$ is in k -duality with \tilde{T} , it is also in k -duality with $\tilde{\underline{T}}$. Since \tilde{T}^* and $\tilde{\underline{T}}^*$ are thus stably conjugate, and since these tori are well-determined only up to such conjugacy, we may, without loss of generality, assume that they are equal.

Given $t \in \tilde{T}_0$ and $\gamma \in \Gamma$, we have that

$$\begin{aligned} {}^h[(\varphi(\gamma))(g t)] &= {}^{hg}[(\varphi(\gamma))(t)] && \text{since } g \text{ is fixed by } \varphi(\Gamma) \\ &= \underline{g}[(\varphi(\gamma))(t)] && \text{since } \underline{\varphi}(\gamma) \text{ and } \varphi(\gamma) \text{ agree on } T_0 \\ &= (\underline{\varphi}(\gamma))(\underline{g}t) && \text{since } \underline{g} \text{ is fixed by } \underline{\varphi}(\Gamma) \\ &= (\underline{\varphi}(\gamma))({}^h({}^g t)), \end{aligned}$$

so $\text{Int}(h) \circ \varphi(\gamma) = \underline{\varphi}(\gamma) \circ \text{Int}(h)$ on \tilde{T} . Therefore,

$$(7.2) \quad \text{Int}(h) \circ \mathcal{N} = \underline{\mathcal{N}} \circ \text{Int}(h),$$

where $\mathcal{N}: \tilde{T} \rightarrow T$ and $\underline{\mathcal{N}}: \tilde{\underline{T}} \rightarrow \underline{T}$ are the norm maps defined with respect to φ and $\underline{\varphi}$, respectively

As in [1, §7], we have an isomorphism

$$\tilde{\delta}: \mathbf{X}^*(\tilde{T}) \rightarrow \mathbf{X}_*(\tilde{T}^*) \quad (\text{resp. } \delta: \mathbf{X}^*(T) \rightarrow \mathbf{X}_*(T^*))$$

that implements k -duality between \tilde{T} and \tilde{T}^* (resp. T and T^*). Let

$$\underline{\delta}: \mathbf{X}^*(\underline{\tilde{T}}) \rightarrow \mathbf{X}_*(\underline{\tilde{T}}^*) \quad (\text{resp. } \underline{\delta}: \mathbf{X}^*(\underline{T}) \rightarrow \mathbf{X}_*(\underline{T}^*))$$

be the analogous isomorphism implementing duality between $\underline{\tilde{T}}$ and $\underline{\tilde{T}}^*$ (resp. \underline{T} and \underline{T}^*). By [1, Remark 6.6], we may choose $\tilde{\delta}, \underline{\delta}$ such that for $\tilde{\chi} \in \mathbf{X}^*(\tilde{T})$ and $\chi \in \mathbf{X}^*(T)$, we have

$$(7.3) \quad \tilde{\delta}({}^h \tilde{\chi}) = \tilde{\delta}(\tilde{\chi}) \quad \underline{\delta}({}^h \chi) = \delta(\chi).$$

Recall also that we have maps

$$\mathcal{N}^* = \mathcal{N}_T^*: \mathbf{X}^*(T) \rightarrow \mathbf{X}^*(\tilde{T}), \quad \underline{\mathcal{N}}^* = \underline{\mathcal{N}}_{\underline{T}}^*: \mathbf{X}^*(\underline{T}) \rightarrow \mathbf{X}^*(\underline{\tilde{T}})$$

dual to the norm maps above, and that we can use the above duality maps to define maps $\hat{\mathcal{N}}_* = \hat{\mathcal{N}}_{T^*,*}$ and $\hat{\underline{\mathcal{N}}}_* = \hat{\underline{\mathcal{N}}}_{\underline{T}^*,*}$ from $\mathbf{X}_*(T^*)$ to $\mathbf{X}_*(\tilde{T}^*)$. To prove (c), it suffices to show that $\hat{\mathcal{N}}_{T^*}(s) = \hat{\underline{\mathcal{N}}}_{\underline{T}^*}(s)$.

Now consider the following diagram:

$$\begin{array}{ccccc}
 & & \mathbf{X}^*(\tilde{T}) & \xrightarrow[\sim]{\tilde{\delta}} & \mathbf{X}_*(\tilde{T}^*) \\
 & \nearrow \sim & \uparrow \text{Int}(h) & & \nearrow \sim \\
 \mathbf{X}^*(\tilde{T}) & \xrightarrow[\sim]{\tilde{\delta}} & \mathbf{X}_*(\tilde{T}^*) & & \uparrow \hat{\mathcal{N}}_* \\
 & \downarrow \mathcal{N}^* & & & \\
 & \mathbf{X}^*(T) & \xrightarrow[\sim]{\delta} & \mathbf{X}_*(T^*) & \\
 & \nearrow \sim & \uparrow \text{Int}(h) & & \nearrow \sim \\
 \mathbf{X}^*(T) & \xrightarrow[\sim]{\delta} & \mathbf{X}_*(T^*) & & \uparrow \hat{\mathcal{N}}_* \\
 & & & & \uparrow \hat{\mathcal{N}}_*
 \end{array}$$

From (7.2), the left-hand face commutes. From (7.3), so do the top and bottom faces. The front and back faces commute by the definitions of $\hat{\mathcal{N}}_*$ and $\hat{\mathcal{N}}_*$. Therefore, the right-hand face commutes. That is, $\hat{\mathcal{N}}_* = \hat{\mathcal{N}}_*$, hence $\hat{\mathcal{N}}_{T^*}(s) = \hat{\mathcal{N}}_{T^*}(s)$, completing the proof. \square

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